

UNIFORMIZATION OF SPHERICAL CR MANIFOLDS

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ABSTRACT. Let M be a closed (compact with no boundary) spherical CR manifold of dimension $2n + 1$. Let \widetilde{M} be the universal covering of M . Let Φ denote a CR developing map

$$\Phi : \widetilde{M} \rightarrow S^{2n+1}$$

where S^{2n+1} is the standard unit sphere in complex $n + 1$ -space C^{n+1} . Suppose that the CR Yamabe invariant of M is positive. Then we show that Φ is injective for $n \geq 3$. In the case $n = 2$, we also show that Φ is injective under the condition: $s(M) < 1$. It then follows that M is uniformizable.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Spherical CR structures are modeled on the boundary of complex hyperbolic space. There have been many studies in various aspects for this structure (e.g., [3], [18], [11], [14], [9], [24]). In this paper, we study the uniformization problem. Let S^{2n+1} denote the standard unit sphere in complex $n + 1$ -space C^{n+1} . Let us start with the CR automorphism group $Aut_{CR}(S^{2n+1})$ of S^{2n+1} , which is the group of complex fractional linear transformation $SU(n + 1, 1)/(\text{center})$. We have the following complex analogue of the Liouville theorem in conformal geometry ([10]).

Lemma 1.1. Let f be a CR diffeomorphism from a connected open set U in S^{2n+1} . If $f(U) \subset S^{2n+1}$, then f is the restriction to U of a complex fractional linear transformation.

Let M be a spherical CR manifold of dimension $2n + 1$. Let \widetilde{M} be the universal covering of M . Using analytic continuation and Lemma 1.1, we get a CR immersion $\Phi : \widetilde{M} \rightarrow S^{2n+1}$. The map Φ is unique up to composition with elements of $Aut_{CR}(S^{2n+1})$ acting on S^{2n+1} . Such a map Φ is called a CR developing map.

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We will determine when Φ is injective. Let $\lambda(M)$ denote the CR Yamabe invariant of M (see (2.10) in Section 2). In the case of $n = 2$, we also need a condition on another CR invariant $s(M)$ which measures the integrability of a positive minimal Green's function G_p on \widetilde{M} (see Theorem 3.4 in Section 3):

$$s(M) := \inf \left\{ s : \int_{\widetilde{M} \setminus U_p} G_p^s dV_\theta < \infty \right\}$$

where U_p is a neighborhood of p . (see (3.6) in Section 3). Observe that $s(M) \leq 1$ in general (see Theorem 3.5 in Section 3). We have the following result.

Theorem A. *Let M be a closed (compact with no boundary) spherical CR manifold of dimension $2n + 1$ with $\lambda(M) > 0$. Let \widetilde{M} be the universal covering of M . Let Φ denote a CR developing map*

$$\Phi : \widetilde{M} \rightarrow S^{2n+1}$$

Then Φ is injective for $n \geq 3$. In case $n = 2$, Φ is injective if we further assume $s(M) < 1$.

Theorem A implies that a closed spherical CR manifold M with $\lambda(M) > 0$ is uniformizable. Let $\pi_1(M)$ denote the fundamental group of M . The CR developing map Φ induces a group homomorphism:

$$\Phi_* : \pi_1(M) \rightarrow \text{Aut}_{CR}(S^{2n+1}).$$

In case Φ is injective, the group homomorphism Φ_* is injective. Note that $\pi_1(M)$ acts on \widetilde{M} by deck transformations. The following result follows from Theorem A.

Corollary B. *Suppose that we are in the situation of Theorem A. Then M is CR diffeomorphic to the quotient Ω/Γ where $\Omega = \Phi(\widetilde{M}) \subset S^{2n+1}$ and $\Gamma = \Phi_*(\pi_1(M))$ for $n \geq 2$. Moreover, Γ is a discrete subgroup of $\text{Aut}_{CR}(S^{2n+1})$ and acts on Ω properly discontinuously.*

The idea of the proof of Theorem A follows a similar line as for the conformal case. Basically we will be dealing with the Green's functions of the CR invariant sublaplacian (see (2.7) in Section 2) on different spaces. In particular, the idea of comparing the pull-back \tilde{G} of the Green's function on S^{2n+1} with the (minimal positive) Green's function G of \widetilde{M} follows the work of Schoen and Yau ([26] or [27]). We reduce the injectivity problem to the estimate of the quotient $v := \frac{\tilde{G}}{G}$. As expected,

the CR Bochner formula for v contains an extra cross term which has no Riemannian analogue. Fortunately we can manage this extra cross term by converting it into a term involving a Paneitz-like operator P (see (2.3) in Section 2). The nonnegativity of P for $n \geq 2$ (see line 5 in the proof of Proposition 3.2 in [13] or [5]) helps to simplify the estimates (see (4.13) and (4.36) in Section 4). We can finally prove $v = 1$, and hence Φ is injective under the condition mentioned in Theorem A.

There has been an unpublished paper ([22]) about this uniformization problem, circulating for years. The main difference between our paper and [22] is the treatment of the CR Bochner formula. We have realized the important role of that Paneitz-like operator P in the CR setting of the Bochner formula through the study of some other problems in recent years (e.g., [4], [5]). So we can clarify some estimates in [22] and conclude a new result in the case of $n = 2$.

Based on the uniformization of spherical CR manifolds, in his another unpublished paper ([23]), using Bony's strong maximum principle, Z. Li showed the nonnegativity of the CR mass (see Definition 5.1 in Section 5). We state his result as Corollary C:

Corollary C. *Let M be a closed spherical CR manifold with $\lambda(M) > 0$. Then, for $n \geq 3$, the CR mass $A_b > 0$ unless M is the standard sphere. In case $n = 2$, the same result also holds if we assume further $s(M) < 1$.*

REMARK 1.1. Notice that Z. Li's arguments are valid, provided that the CR developing map Φ is injective. We rewrite his proof in Section 5. To solve the CR Yamabe problem by producing a minimizer for the Yamabe (or Tanaka-Webster) quotient, one needs to work out a test function estimate by using the above positive mass theorem. This has been done in [23] (see also unpublished notes of Andrea Malchiodi). In dimension 3, we also have a positive mass theorem under the condition $P \geq 0$ through a different approach ([8]). Note that in dimension 3 (with $\lambda(M) > 0$), the condition $P \geq 0$ is not automatic and is shown to be almost equivalent to the embeddability of the underlying CR structure ([6], [7]).

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2. BOCHNER FORMULAS AND CR INVARIANT OPERATORS

Let M be a smooth (meaning C^∞ throughout the paper) $(2n+1)$ -dimensional (paracompact) manifold. A contact structure or bundle ξ on M is a completely nonintegrable $2n$ -dimensional distribution. A contact form is a 1-form annihilating ξ . Let (M, ξ) be a contact $(2n+1)$ -dimensional manifold with an oriented contact structure ξ . There always exists a global oriented contact form θ , obtained by patching together local ones with a partition of unity. The Reeb vector field of θ is the unique vector field T such that $\theta(T) = 1$ and $\mathcal{L}_T\theta = 0$ or $d\theta(T, \cdot) = 0$. A CR -structure compatible with ξ is a smooth endomorphism $J : \xi \rightarrow \xi$ such that $J^2 = -\text{Identity}$. Let $T_{1,0} \subset \xi \otimes \mathbb{C}$ denote the n -dimensional complex subbundle of $TM \otimes \mathbb{C}$, consisting of eigenvectors of J with eigenvalue i . We will assume throughout that the CR structure J is integrable, that is, $T_{1,0}$ satisfies the condition $[T_{1,0}, T_{1,0}] \subset T_{1,0}$. A pseudohermitian structure compatible with an oriented contact structure ξ is a CR -structure J compatible with ξ together with a global contact form θ . On ξ , we define the Levi form $L_\theta := \frac{1}{2}d\theta(\cdot, J\cdot)$. If L_θ is definite (independent of the choice of contact form), M (or (M, ξ, J)) is said to be strictly pseudoconvex. We call θ positive if L_θ is positive definite (often called Levi metric in this case). We will always assume that M is strictly pseudoconvex and θ is positive.

Given a pseudohermitian structure (J, θ) (with J integrable and θ positive), we can choose complex vector field Z_α , $\alpha = 1, 2, \dots, n$, eigenvectors of J with eigenvalue i , and complex 1-form θ^α , $\alpha = 1, 2, \dots, n$, such that $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$ is dual to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ ($\theta^{\bar{\alpha}} = (\theta^\alpha)^\flat$, $Z_{\bar{\alpha}} = (Z_\alpha)^\sharp$). It follows that

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}$$

(summation convention throughout) for some hermitian matrix of functions $(h_{\alpha\bar{\beta}})$, which is positive definite since M is strictly pseudoconvex and θ is positive.

The pseudohermitian connection of (J, θ) is the connection $\nabla^{p.h.}$ on $TM \otimes \mathbb{C}$ (and extended to tensors) given by

$$\nabla^{p.h.} Z_\alpha = \omega_\alpha{}^\beta \otimes Z_\beta, \nabla^{p.h.} Z_{\bar{\alpha}} = \omega_{\bar{\alpha}}{}^{\bar{\beta}} \otimes Z_{\bar{\beta}}, \nabla^{p.h.} T = 0$$

in which the 1-forms $\omega_\alpha{}^\beta$ are uniquely determined by the following equations with a normalization condition ([29], [28], [19]):

$$\begin{aligned} (2.1) \quad d\theta^\beta &= \theta^\alpha \wedge \omega_\alpha{}^\beta + A^\beta{}_{\bar{\alpha}} \theta \wedge \theta^{\bar{\alpha}}, \\ dh_{\alpha\bar{\gamma}} &= \omega_\alpha{}^\beta h_{\beta\bar{\gamma}} + h_{\alpha\bar{\beta}} \omega_{\bar{\gamma}}{}^{\bar{\beta}}. \end{aligned}$$

The coefficient $A_{\bar{\alpha}}^{\beta}$ in (2.1) is called the (pseudohermitian) torsion. As usual we use the matrix $h_{\alpha\bar{\beta}}$ to raise and lower indices, e.g., $A_{\alpha\gamma} = h_{\alpha\bar{\beta}}A_{\bar{\beta}}^{\beta}\gamma$ where $A_{\bar{\beta}}^{\beta}\gamma$ is the complex conjugate of $A_{\bar{\gamma}}^{\beta}$. We define covariant differentiation with respect to the connection $\nabla^{p.h.}$. For a real C^∞ smooth function φ , we have $\varphi_0 := T\varphi$, $\varphi_\alpha := Z_\alpha\varphi$, $\varphi_{\alpha\beta} := Z_\beta Z_\alpha\varphi - \omega_\alpha^\gamma(Z_\beta)Z_\gamma\varphi$, etc.. For the subgradient $\nabla_b\varphi$, the sublaplacian $\Delta_b\varphi$, and the subhessian $(\nabla^H)^2\varphi$, we have the following formulas:

$$\begin{aligned}\nabla_b\varphi &= \varphi^\alpha Z_\alpha + \varphi^{\bar{\alpha}} Z_{\bar{\alpha}} \\ \Delta_b\varphi &= -(\varphi_\alpha^\alpha + \varphi_{\bar{\alpha}}^{\bar{\alpha}}) \\ (\nabla^H)^2\varphi &= \varphi_\alpha^\beta \theta^\alpha \otimes Z_\beta + \varphi_{\bar{\alpha}}^{\bar{\beta}} \theta^{\bar{\alpha}} \otimes Z_{\bar{\beta}} \\ &\quad + \varphi_\alpha^{\bar{\beta}} \theta^\alpha \otimes Z_{\bar{\beta}} + \varphi_{\bar{\alpha}}^{\beta} \theta^{\bar{\alpha}} \otimes Z_{\bar{\beta}}.\end{aligned}$$

Differentiating ω_β^α gives

$$\begin{aligned}d\omega_\beta^\alpha - \omega_\beta^\gamma \wedge \omega_\gamma^\alpha \\ = R_{\beta}^\alpha{}_{\rho\bar{\sigma}}\theta^\rho \wedge \theta^{\bar{\sigma}} + iA_{\bar{\gamma}}^\alpha\theta_\beta \wedge \theta^{\bar{\gamma}} - iA_{\beta\gamma}\theta^\gamma \wedge \theta^\alpha \text{ mod } \theta\end{aligned}$$

where $R_{\beta}^\alpha{}_{\rho\bar{\sigma}}$ is the Tanaka-Webster curvature. Write $R_{\alpha\bar{\beta}} := R_\gamma^\gamma{}_{\alpha\bar{\beta}}$ and $R := R_\alpha^\alpha$. For $X = X^\alpha Z_\alpha$, $Y = Y^\beta Z_\beta \in T_{1,0}$, we define

$$\begin{aligned}Ric(X, Y) &= R_{\alpha\bar{\beta}}X^\alpha Y^{\bar{\beta}} \\ Tor(X, Y) &= 2\text{Re}(iA_{\bar{\alpha}\bar{\beta}}X^{\bar{\alpha}}Y^{\bar{\beta}}).\end{aligned}$$

We recall the pointwise Bochner formula ([15]):

$$\begin{aligned}(2.2) \quad \frac{1}{2}\Delta_b|\nabla_b\varphi|^2 &= -|(\nabla^H)^2\varphi|^2 + \langle \nabla_b\varphi, \nabla_b\Delta_b\varphi \rangle \\ &\quad - 2Ric((\nabla_b\varphi)_C, (\nabla_b\varphi)_C) \\ &\quad + (n-2)Tor((\nabla_b\varphi)_C, (\nabla_b\varphi)_C) \\ -2 &< J\nabla_b\varphi, \nabla_b\varphi_0 >\end{aligned}$$

for a real smooth function φ , where the length $|\cdot|$ and the inner product $\langle \cdot, \cdot \rangle$ are with respect to the Levi metric L_θ and $(\nabla_b\varphi)_C := \varphi^\alpha Z_\alpha$.

We define a Paneitz-like operator P by

$$(2.3) \quad P\varphi := 4(\varphi_{\bar{\alpha}}^{\bar{\alpha}}{}_\beta + inA_{\beta\alpha}\varphi^\alpha)^\beta + \text{conjugate}.$$

Let $P_\beta\varphi := \varphi_{\bar{\alpha}}^{\bar{\alpha}}{}_\beta + inA_{\beta\alpha}\varphi^\alpha$. For $n = 1$, the CR pluriharmonic functions are characterized by $P_1\varphi = 0$ ([20]) while, for $n \geq 2$, they are characterized by $P\varphi = 0$. (see [13] in which P is also identified with the compatibility operator for solving a certain degenerate Laplace equation in the case of $n = 1$). On the other hand, this operator P is a CR analogue of the Paneitz operator in conformal geometry (see [16])

for the relation to a CR analogue of the Q -curvature and the log-term coefficient in the Szegő kernel expansion). On a closed pseudohermitian $(2n+1)$ -dimensional manifold (M, J, θ) , we call P nonnegative if there holds

$$(2.4) \quad \int_M \varphi(P\varphi) dV_\theta \geq 0$$

for all real smooth functions φ , in which the volume form $dV_\theta = \theta \wedge (d\theta)^n$. We need the integrated Bochner formula:

$$(2.5) \quad \begin{aligned} \int_M \varphi_0^2 dV_\theta &= \frac{1}{n^2} \int_M (\Delta_b \varphi)^2 dV_\theta \\ &\quad + \frac{2}{n} \int_M \text{Tor}((\nabla_b \varphi)_C, (\nabla_b \varphi)_C) dV_\theta \\ &\quad - \frac{1}{2n^2} \int_M \varphi(P\varphi) dV_\theta \end{aligned}$$

(see Corollary 2.4 in [5]). By Theorem 3.2 in [5], we learn that P is nonnegative for $n \geq 2$. It follows from (2.5) and (2.4) that

$$(2.6) \quad \begin{aligned} \int_M \varphi_0^2 dV_\theta &\leq \frac{1}{n^2} \int_M (\Delta_b \varphi)^2 dV_\theta \\ &\quad + 2\kappa \int_M |\nabla_b \varphi|^2 dV_\theta \end{aligned}$$

where $\kappa := \max_{q \in M} (\sum_{\alpha, \beta} (A_{\alpha\beta} A^{\alpha\beta})(q))^{1/2}$ (note that $\sum_{\alpha, \beta} A_{\alpha\beta} A^{\alpha\beta}$ is independent of the choice of frames and is a nonnegative real function on M).

Let $b_n := 2 + \frac{2}{n}$. We define the CR invariant sublaplacian D_θ by

$$(2.7) \quad D_\theta = b_n \Delta_b + R$$

where R denotes the Tanaka-Webster scalar curvature (with respect to θ while fixing J). Suppose that $\tilde{\theta} = u^{\frac{2}{n}} \theta$ for a positive C^∞ smooth function u . Then for any real smooth function φ , there holds

$$(2.8) \quad D_\theta(u\varphi) = u^{1+\frac{2}{n}} D_{\tilde{\theta}}(\varphi).$$

Letting $\varphi \equiv 1$ in (2.8) gives the transformation law for R :

$$\tilde{R} = u^{-1-\frac{2}{n}} D_\theta(u)$$

where \tilde{R} denotes the Tanaka-Webster scalar curvature with respect to $\tilde{\theta}$. The Yamabe problem on a CR manifold is to find u (or $\tilde{\theta}$) such

that \tilde{R} is a given constant. This is the Euler-Lagrange equation of the following energy functional:

$$E_\theta(u) := \int_M (b_n |\nabla_b u|^2 + Ru^2) dV_\theta.$$

for positive smooth functions u such that

$$(2.9) \quad \int_M |u|^{b_n} dV_\theta = 1.$$

The CR Yamabe invariant $\lambda(M)$ has the following expression:

$$(2.10) \quad \lambda(M) = \inf_{u \in \Xi} E_\theta(u)$$

where Ξ is the space of positive smooth (with compact support if M is noncompact) functions u satisfying (2.9). For M closed, it is known that $\lambda(M) > 0$ is equivalent to the existence of a contact form $\bar{\theta}$ with respect to which $\bar{R} > 0$.

Let (M, J, θ) be a closed pseudohermitian manifold with $R > 0$. Let $\Gamma^\beta(M)$ denote the nonisotropic Hölder space of exponent β (page 181 in [21] or [12] for the local description modelled on the Heisenberg group). Following a standard argument in [1] for the elliptic case, we obtain

Proposition 2.1. *Let (M, J, θ) be a closed pseudohermitian manifold with $R > 0$. Then for any $f \in \Gamma^\beta(M)$, β a noninteger > 0 , there exists a unique $u \in \Gamma^{\beta+2}(M)$ such that*

$$D_\theta(u) = f.$$

Using Proposition 2.1 and a similar construction in [1], we have that for any $p \in M$, there is a unique Green's function G_p for D_θ with pole at p .

3. THE GREEN'S FUNCTION OF THE UNIVERSAL COVERING

Let S^{2n+1} denote the unit sphere in C^{n+1} . Let $\hat{\xi} := TS^{2n+1} \cap J_{C^{n+1}}(TS^{2n+1})$ be the standard contact bundle over S^{2n+1} , where $J_{C^{n+1}}$ denotes the almost complex structure of C^{n+1} . Let \hat{J} be the restriction of $J_{C^{n+1}}$ to $\hat{\xi}$. We call a CR manifold (M, J) (or (M, ξ, J) , resp.) spherical if it is locally CR equivalent to (S^{2n+1}, \hat{J}) (or $(S^{2n+1}, \hat{\xi}, \hat{J})$, resp.) (cf. e.g., [3]). Let (M, J, θ) be a closed pseudohermitian manifold of dimension $2n + 1$ with (M, J) spherical and $R > 0$. Let \widetilde{M} be the

universal covering of M with the CR structure π^*J and the contact form $\pi^*\theta$, where

$$\pi : \widetilde{M} \rightarrow M$$

is the canonical projection. It follows that \widetilde{M} has no boundary. If \widetilde{M} is compact, then (\widetilde{M}, π^*J) must be CR equivalent to (S^{2n+1}, \hat{J}) since it is simply connected and spherical. We will assume that \widetilde{M} is noncompact (or $\pi_1(M)$ is an infinite group) throughout this section. We will still use θ to mean $\pi^*\theta$. Our goal in this section is to study the existence of the Green's function for D_θ on \widetilde{M} and its decay property at the geometric boundary of \widetilde{M} .

Let Ω be a relatively compact smooth domain in \widetilde{M} and $p \in \Omega$. We would like to construct the Dirichlet Green's function G_p^Ω for the domain Ω , that is, to prove the following

Theorem 3.1. *There exists a unique $G_p^\Omega \in C^\infty(\Omega \setminus \{p\}) \cap C(\overline{\Omega} \setminus \{p\})$ such that*

$$(3.1) \quad \begin{aligned} D_\theta(G_p^\Omega) &= \delta_p \text{ in } \Omega \\ G_p^\Omega|_{\partial\Omega} &= 0 \end{aligned}$$

Once G_p^Ω is constructed, the symmetry property $G_p^\Omega(q) = G_q^\Omega(p)$ is due to the fact that D_θ is self-adjoint and from the integration by parts argument as in the elliptic case. The positivity of G_p^Ω is due to the fact that the leading order operator of D_θ is nonnegative and the Tanaka-Webster scalar curvature R is positive.

As in the elliptic case, the existence of the Dirichlet Green's function is equivalent to the solvability of nonhomogeneous Dirichlet problem with zero boundary value. So solving (3.1) is reduced to solving the following Dirichlet problem..

Theorem 3.2. *Let Ω be a relatively compact smooth domain in \widetilde{M} and $f \in \Gamma^\beta(\overline{\Omega})$. Then there is a unique $u \in \Gamma^{\beta+2}(\Omega) \cap C(\overline{\Omega})$ such that*

$$(3.2) \quad \begin{aligned} D_\theta(u) &= f \text{ in } \Omega \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

The uniqueness of the solution to (3.2) follows from the following lemma.

Lemma 3.3. *Let Ω be a relatively compact smooth domain in \widetilde{M} such that for $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$, there holds*

$$\begin{aligned} D_\theta(u) &\leq D_\theta(v) \text{ in } \Omega \\ u|_{\partial\Omega} &= v|_{\partial\Omega}. \end{aligned}$$

Then $u < v$ in Ω unless $u = v$ in Ω .

To prove Lemma 3.3, we observe that the leading order part of D_θ is a subelliptic operator of Hormander's type (sum of square vector fields). Then one can apply Bony's arguments without essential change by the local nature of this lemma. To prove Theorem 3.2, we use Perron's construction which relies heavily on the maximum principle and the solvability of the Dirichlet problem for the balls. We remark that the key step of proving Theorem 3.2 is to localize the problem. In this respect, the CR invariance enables us to reduce the problem on (\widetilde{M}, θ) (J omitted) for D_θ to the problem on the Heisenberg group (H^n, Θ) for D_Θ ([21]).

We will often call a Heisenberg ball simply a ball in this section. Let B denote a small ball in \widetilde{M} , identified with a Heisenberg ball in H^n . There is a positive function $\phi \in C^\infty(\widetilde{M})$ such that $\phi^{\frac{2}{n}}\theta = \Theta$ in B . By the known results on H^n and the transformation law for D_θ , we conclude that for each $f \in \Gamma^\beta(\bar{B})$ and $g \in C(\partial B)$, there is a unique $u \in \Gamma^{\beta+2}(B) \cap C(\bar{B})$ such that

$$\begin{aligned} D_\theta(u) &= f \text{ in } B \\ u|_{\partial B} &= g \end{aligned}$$

In the Dirichlet problem for D_θ on a smooth domain in \widetilde{M} , the question of continuity up to the boundary is a purely local issue. So we will deal with it in the same localizing spirit as above.

We begin the Perron process by generalizing the notion of subsolution in classical elliptic theory to the operator D_θ on a smooth domain Ω in \widetilde{M} . Note that D_θ has a nonnegative leading order part.

Definition. A continuous function u in Ω is called a subsolution to the equation

$$D_\theta(v) = f$$

where $f \in \Gamma^\beta(\bar{\Omega})$, β a noninteger > 0 , if for every ball $B \subset\subset \Omega$ and v such that

$$D_\theta(v) = f \text{ in } B, \quad u \leq v \text{ on } \partial B,$$

then we have that $u \leq v$ in B .

Analogously we can define the notion of supersolution as well. These notions are completely in parallel to the notions of continuous subharmonic and superharmonic functions in classical elliptic theory. The significance of these notions are ensured by the Bony's maximum principle. They also have the following useful properties of sub and supersolutions:

- (1) If $u \in C^2(\Omega)$, then u is a subsolution (supersolution, resp.) if and only if that $D_\theta(u) \leq f$ ($D_\theta(u) \geq f$, resp.).
- (2) If u_1, \dots, u_m are subsolutions (supersolutions, resp.) in Ω , then $\max\{u_j : 1 \leq j \leq m\}$ ($\min\{u_j : 1 \leq j \leq m\}$, resp.) is also a subsolution (supersolution, resp.) in Ω .
- (3) Suppose that $B \subset\subset \Omega$ and u_1 satisfies

$$\begin{aligned} D_\theta(u_1) &= f \text{ in } B \\ u_1 &= u_2 \text{ in } \partial B \end{aligned}$$

where u_2 is a subsolution in Ω . Then

$$u = \begin{cases} u_1 & \text{in } B \\ u_2 & \text{in } \Omega \setminus B \end{cases}$$

is also a subsolution in Ω .

Proof. (of Theorem 3.2) We will carry out the proof in the spirit of standard arguments in the elliptic theory. It consists of two main steps:

Step 1 : Construction of the Perron solution. Consider the following set of subsolutions

$$S = \{v : v \text{ is a subsolution for } D_\theta(w) = f, v|_{\partial\Omega} \leq 0\}$$

Note that the Tanaka-Webster scalar curvature on \widetilde{M} has a positive lower bound, say R_0 . We observe that $\frac{-\sup|f|}{R_0} \in S$ and the constant $\frac{\sup|f|}{R_0}$ is a supersolution. Therefore $u(x) = \sup_{v \in S} v(x)$ is well defined.

We would like to show that u is the Perron solution, i.e. $D_\theta u = f$ in Ω . Let $p \in \Omega$ be an arbitrary fixed point. By the definition of u , there exists a sequence $v_m \in S$ such that $v_m(p) \rightarrow u(p)$. By replacing v_m with $\max\{v_1, \dots, v_m\}$, we may assume that the sequence is monotone.

Now choose a ball $B \subset\subset \Omega$ of p such that the geometry of B can be flattened after a conformal change of a contact form. Let w_m be the unique solution satisfying

$$\begin{aligned} D_\theta(w_m) &= f \text{ in } B \\ w_m &= v_m \text{ in } \partial B. \end{aligned}$$

It follows from earlier discussion that

$$W_m = \begin{cases} w_m & \text{in } B \\ v_m & \text{in } \Omega \setminus B \end{cases}$$

is a subsolution in Ω . Hence, $u(p) \geq W_m(p) \geq v_m(p) \rightarrow u(p)$. Because W_m is monotone increasing in B , the limit $W = \lim_{m \rightarrow \infty} W_m$ exists in B . As in the elliptic theory, the subelliptic apriori estimates imply that the sequence W_m contains a subsequence converging in B , and hence W is a solution in B and $W(p) = u(p)$.

We claim that $W = u$ in B to complete step 1. The arguments are standard: let $q \in B$, there is a monotone increasing sequence $g_m \in S$ such that $g_q \rightarrow u(q)$. Let h_m solve the equation

$$\begin{aligned} D_\theta(w) &= f & \text{in } B \\ w &= \bar{g}_m & \text{in } \partial B \end{aligned}$$

where in B , $\bar{g}_m = \max\{g_m, v_m\}$. Therefore the sequence is also monotone increasing and $v_m \leq h_m$ in B . As before, $h = \lim_{m \rightarrow \infty} h_m$ is a solution in B and $h(q) = u(q)$. Since $u(p) \geq h(p) \geq W(p) = u(p)$, Bony's strong maximum principle implies that $W = h$ in B . Therefore, u is indeed an interior solution.

Step 2 : Continuity up to the boundary. Fix $p \in \partial\Omega$, we choose a small ball B of p such that the boundary of $B \cap \Omega$ is smooth and the geometry of B can be flattened by choosing a conformal contact form. Notice the following two facts:

(1) There exists u_f which solves the Dirichlet problem:

$$\begin{aligned} D_\theta(u_f) &= f & \text{in } B \cap \Omega \\ u_f &= 0 & \text{in } \partial(B \cap \Omega). \end{aligned}$$

(2) The existence of a local barrier at p , that is, for a small ball of p , there is a function $w \in C(\overline{B} \cap \overline{\Omega}) \cap C^2(B \cap \Omega)$ such that

$$D_\theta(w) = 0 \text{ in } B \cap \Omega, w > 0 \text{ in } \overline{B} \cap \overline{\Omega} \setminus \{p\} \text{ and } w(p) = 0.$$

We will use this local barrier to construct a global barrier and show that the Perron solution u obtained in step 1 is continuous up to the boundary. For any ε , there exists δ and K such that

$$|u_f(x)| \leq \varepsilon \text{ if } |x - p| \leq \delta$$

and

$$Kw(x) \geq \sup u_f \text{ if } |x - p| \geq \delta.$$

We will consider $w_1 = -Kw + u_f - \varepsilon$ in $B \cap \Omega$. Then we have immediately that

$$D_\theta w_1 = f \text{ and } w_1 < 0 \text{ in } B \cap \Omega.$$

Consider the following number

$$M_K = \sup_{\partial(B \cap \Omega) \setminus \partial\Omega} w_1 = \sup_{\partial(B \cap \Omega) \setminus \partial\Omega} (-Kw - \varepsilon) = -K \left(\min_{\partial(B \cap \Omega) \setminus \partial\Omega} w \right) - \varepsilon,$$

and we let

$$W_1 = \begin{cases} \max(w_1, M_K) & \text{in } \overline{B} \cap \overline{\Omega} \\ M_K & \text{in } \overline{\Omega} \setminus \overline{B}. \end{cases}$$

Note that $W_1|_{\partial(B \cap \Omega) \setminus \partial\Omega} = M_K$, so $W_1 \in C(\Omega)$. It is easy to check directly that W_1 is a subsolution (when K is large enough). To check the boundary behavior of W_1 , we note that (when q is very close to p):

$$W_1(q) = \max(W_1(q), M_K) = -Kw(q) + u_f(q) - \varepsilon \rightarrow -\varepsilon$$

as $q \rightarrow p$. So it is zero at p and negative everywhere else on $\partial\Omega$. Therefore, $W_1 \in S$ and W_1 is a global boundary barrier at p . It follows that

$$\liminf_{q \rightarrow p} u(q) \geq \lim_{q \rightarrow p} w_1(q) = -\varepsilon.$$

Since ε is arbitrary, we have the continuity of u up to the boundary. \square

We are now ready to construct a Green's function G_p for D_θ with pole at $p \in \widetilde{M}$. We would also like to discuss its decay properties at the infinity.

Recall that $\Phi : \widetilde{M} \rightarrow S^{2n+1}$ denotes the CR developing map. Let H_y be the Green's function for the CR invariant sublaplacian D_0 of $(S^{2n+1}, \theta_{S^{2n+1}})$ with the pole $y = \Phi(p)$, where $\theta_{S^{2n+1}}$ is the standard contact form on S^{2n+1} . Since Φ is a CR immersion, we can write

$$\Phi^*(\theta_{S^{2n+1}}) = |\Phi'|^2 \theta$$

where θ is the contact form for \widetilde{M} and $|\Phi'|$ is a positive C^∞ smooth function on \widetilde{M} . By the transformation law (2.8) of the CR invariant sublaplacian, we immediately obtain the following formula

$$(3.3) \quad D_\theta(|\Phi'|^n H_y \circ \Phi) = \sum_{\bar{p} \in \Phi^{-1}(y)} |\Phi'(\bar{p})|^{n+2} \delta_{\bar{p}}.$$

Let us consider the following function (with poles in $\Phi^{-1}(y)$):

$$(3.4) \quad \overline{G} := |\Phi'(p)|^{-(n+2)} |\Phi'|^n H_y \circ \Phi.$$

Since Φ is a CR immersion, \overline{G} is positive, C^∞ smooth, and $D_\theta \overline{G} = 0$ on $\widetilde{M} \setminus \Phi^{-1}(y)$. Also, \overline{G} has exactly the same asymptotic behavior at each of $\Phi^{-1}(y)$. We call \overline{G} the normalized pullback of H_y , which will

be taken as a singular barrier in the construction of a global Green's function on \widetilde{M} through a limit procedure of Dirichlet Green's functions.

Let $\{\Omega_k \subset \Omega_{k+1} : k = 1, \dots, \}$ be a relatively compact, C^∞ smooth exhaustion of the universal covering \widetilde{M} . Take $p \in \Omega_1$. Note that

$$D_\theta(\overline{G} - G_p^{\Omega_k}) \geq 0,$$

\overline{G} is positive, and $G_p^{\Omega_k}|_{\partial\Omega_k} = 0$. By the maximum principle of Bony, we see that

$$(3.5) \quad G_p^{\Omega_k} < \overline{G}.$$

away from $\Phi^{-1}(y)$. Near the point p , we have the following equality:

$$D_\theta(\overline{G} - G_p^{\Omega_k}) = 0.$$

So $\overline{G} - G_p^{\Omega_k}$ is smooth near p by the regularity result for Δ_b and D_θ being "covariant" to $D_\Theta = b_n \Delta_b$, Θ : standard contact form in the Heisenberg group (cf. the argument in the end of the proof of Lemma 4.1). Therefore we have the following result.

Theorem 3.4. *Let $G_p = \lim_{k \rightarrow \infty} G_p^{\Omega_k}$. Then G_p is a positive fundamental solution of D_θ with pole at p . Moreover, G_p is minimal among all positive fundamental solutions.*

Proof. By the strong maximum principle of Bony and (3.5), the sequence of the Green's functions $\{G_p^{\Omega_k}\}$ is strictly increasing and has an upper bound. Thus, away from $\Phi^{-1}(y)$, the limit of $G_p^{\Omega_k}$ exists. Next by the standard argument using Bony's maximum principle, $\Phi^{-1}(y) \setminus \{p\}$ is a set of removable singularities for G_p .

The minimality of G_p follows from its construction, i.e., if F_p is another global positive fundamental solution on \widetilde{M} with pole at p , then again Bony's maximum principle implies that $G_p^{\Omega_k} < F_p$ for any k , and the conclusion follows. \square

We would like to discuss the decay properties of the constructed G_p , in particular, its integrability away from the pole p . We define $s(M)$, the minimum exponent of the integrability of G_p by

$$(3.6) \quad s(M) := \inf \left\{ s : \int_{\widetilde{M} \setminus U_p} G_p^s dV_\theta < \infty \right\}$$

where U_p is a neighborhood of p .

Theorem 3.5. *$s(M)$ is a CR invariant and satisfies the following inequality:*

$$(3.7) \quad s(M) \leq 1.$$

Proof. For $p \in \widetilde{M}$, let U_p be a small neighborhood of p with smooth boundary. Let $\{\Omega_k \subset \Omega_{k+1} : k = 1, \dots, \}$ be a relatively compact, smooth exhaustion of the universal covering \widetilde{M} . We may assume that $p \in U_p \subset \Omega_1$. Because the Dirichlet Green's function $G_p^{\Omega_k}$ is smooth in $\Omega_{k-1} \setminus U_p$, Bony's maximum principle implies that

$$\sup_{\Omega_{k-1} \setminus U_p} G_p^{\Omega_k} \leq \sup_{\Omega_k \setminus U_p} G_p^{\Omega_k} \leq \sup_{\partial U_p} G_p^{\Omega_k} < \max_{\partial U_p} G_p$$

Therefore we may (C^∞) smoothly extend $G_p^{\Omega_k}$ into U_p with the extension smaller than $\max_{\partial U_p} G_p$, but positive. Denote this extension (which is a smooth function over Ω_{k-1}) by $\bar{G}_p^{\Omega_k}$.

For each $\alpha \geq 0$, let u_k be the solution to the following Dirichlet problem:

$$\begin{aligned} D_\theta(u) &= (\bar{G}_p^{\Omega_k})^\alpha \text{ in } \Omega_{k-1} \\ u|_{\partial \Omega_{k-1}} &= 0. \end{aligned}$$

By the weak maximum principle, we obtain

$$(3.8) \quad \max_{\Omega_{k-1}} u_k \leq \frac{(\max_{\partial U_p} G_p)^\alpha}{R_0}$$

where $R_0 > 0$ is the lower bound of the Tanaka-Webster scalar curvature of \widetilde{M} . By the solution representation in Ω_{k-1} , we have

$$\begin{aligned} (3.9) \quad u_k(p) &= \int_{\Omega_{k-1}} G_p^{\Omega_{k-1}}(q) (\bar{G}_p^{\Omega_k}(q))^\alpha dV_\theta(q) \\ &\geq \int_{\Omega_{k-1} \setminus U_p} G_p^{\Omega_{k-1}}(q) (\bar{G}_p^{\Omega_k}(q))^\alpha dV_\theta(q) \end{aligned}$$

By the monotonicity of $G_p^{\Omega_k}$ and letting $k \rightarrow \infty$, we conclude that

$$\int_{\widetilde{M} \setminus U_p} G_p^{1+\alpha} dV_\theta \leq \frac{(\max_{\partial U_p} G_p)^\alpha}{R_0}.$$

from (3.9) and (3.8). So (3.7) follows. Finally, we need to show that $s(M)$ is a well defined CR invariant. It is routine to check that the definition of $s(M)$ is independent of the choice of U_p . Also $s(M)$ is independent of the choice of contact form from the transformation law

of Green's functions with respect to two different contact forms (cf. Proposition 2.6 of Chapter VI in [27] for the conformal case).

□

Let ρ denote the Carnot-Carathéodory distance on \widetilde{M} with respect to the Levi metric (see, e.g., [25]). Let $B_r(p) \subset \widetilde{M}$ denote the ball of radius r , centered at p , with respect to the Carnot-Carathéodory distance ρ . From Theorem 3.5 and a Moser's iteration procedure, we have the following result.

Proposition 3.6. *There holds*

$$\lim_{r \rightarrow \infty} (\sup\{G_p(x) : \rho(x, p) \geq r\}) = 0.$$

Proof. Recall that $b_n := 2 + \frac{2}{n}$. By Theorem 3.5, $\int_{\widetilde{M} \setminus U_p} G_p^{b_n} dV_\theta < \infty$. Thus we have

$$\lim_{r \rightarrow \infty} \int_{\{x: \rho(x, p) \geq r\}} G_p^{b_n} dV_\theta = 0.$$

Therefore it is enough to establish the estimate

$$G_p(x) \leq C \left(\int_{B_1(x)} G_p^{b_n} dV_\theta \right)^{1/b_n} \text{ for all } x \in \widetilde{M} \setminus B_2(p)$$

where $B_1(x)$ is a ball of radius 1, centered at x . First, we have the equation for G_p

$$\Delta_b G_p + \frac{1}{b_n} R G_p = 0 \text{ on } \widetilde{M} \setminus B_2(p).$$

Take $q \geq b_n := 2 + \frac{2}{n}$. Multiplying the above formula by $G_p^{q-1} \phi^2$ and integrating by parts give

$$\begin{aligned} (3.10) \quad & (q-1) \int_{\widetilde{M}} \phi^2 G_p^{q-2} |\nabla_b G_p|^2 dV_\theta + \frac{1}{b_n} \int_{\widetilde{M}} R G_p^q \phi^2 dV_\theta \\ & \leq 2 \int_{\widetilde{M}} \phi G_p^{q-1} |\nabla_b \phi| |\nabla_b G_p| dV_\theta \\ & \leq \alpha \int_{\widetilde{M}} \phi^2 G_p^{q-2} |\nabla_b G_p|^2 dV_\theta + \frac{1}{\alpha} \int_{\widetilde{M}} |\nabla_b \phi|^2 G_p^q dV_\theta \end{aligned}$$

for all $\alpha > 0$. Here $\phi \in C_0^\infty(\widetilde{M} \setminus B(p))$. Taking $\alpha = q - 2$ in (3.10), we get

$$(3.11) \quad \int_{\widetilde{M}} \phi^2 G_p^{q-2} |\nabla_b G_p|^2 dV_\theta \leq \frac{1}{q-2} \int_{\widetilde{M}} |\nabla_b \phi|^2 G_p^q dV_\theta$$

by $R > 0$ and $G_p > 0$. On the other hand, taking $\alpha = 1$ in (3.10) gives

(3.12)

$$\begin{aligned} 2 \int_{\widetilde{M}} \phi G_p^{q-1} |\nabla_b \phi| |\nabla_b G_p| dV_\theta &\leq \int_{\widetilde{M}} \phi^2 G_p^{q-2} |\nabla_b G_p|^2 dV_\theta + \int_{\widetilde{M}} |\nabla_b \phi|^2 G_p^q dV_\theta \\ &\leq \left(\frac{q-1}{q-2} \right) \int_{\widetilde{M}} |\nabla_b \phi|^2 G_p^q dV_\theta. \end{aligned}$$

by (3.11). It then follows from (3.12) that

(3.13)

$$\begin{aligned} &\int_{\widetilde{M}} \left(|\nabla_b(\phi G_p^{q/2})|^2 + \frac{1}{b_n} R \phi^2 G_p^q \right) dV_\theta \\ &\leq \int_{\widetilde{M}} \left(|\nabla_b \phi|^2 G_p^q + \frac{q^2}{4} \phi^2 G_p^{q-2} |\nabla_b G_p|^2 + q \phi G_p^{q-1} |\nabla_b \phi| |\nabla_b G_p| + \frac{1}{b_n} R \phi^2 G_p^q \right) dV_\theta \\ &\leq C_n q^2 \int_{\widetilde{M}} (|\nabla_b \phi|^2 + \phi^2) G_p^q dV_\theta \end{aligned}$$

for some constant C_n independent of q . Applying the Sobolev inequality

$$\left(\int_{\widetilde{M}} |\phi|^{b_n} dV_\theta \right)^{2/b_n} \leq \lambda(\widetilde{M})^{-1} \int_{\widetilde{M}} \left(|\nabla_b \phi|^2 + \frac{1}{b_n} R \phi^2 \right) dV_\theta$$

(note that $\lambda(\widetilde{M}) > 0$. In fact, $\lambda(\widetilde{M}) = \lambda(S^{2n+1})$ a CR analogue of Theorem 2.2 of Chapter VI in [27]), we obtain

$$\begin{aligned} &\left[\int_{\widetilde{M}} (\phi G_p^{q/2})^{b_n} \right]^{2/b_n} \\ (3.14) \quad &\leq \lambda(\widetilde{M})^{-1} \int_{\widetilde{M}} \left(|\nabla_b(\phi G_p^{q/2})|^2 + \frac{1}{b_n} R \phi^2 G_p^q \right) dV_\theta \\ &\leq \tilde{C}_n q^2 \int_{\widetilde{M}} (|\nabla_b \phi|^2 + \phi^2) G_p^q dV_\theta \end{aligned}$$

by (3.13) for some constant \tilde{C}_n independent of q .

We will use (3.14) repeatedly with

$$q_0 = b_n = 2r, \quad q_k = q_0 r^k, \quad \text{with } r = \frac{n+1}{n}.$$

Define a sequence of cut-off functions as follows. Set $a_0 = 1$, $a_k = 1 - \sum_{i=1}^k 3^{-i}$ for $k \geq 1$, and we require that for each k the function $\phi_k \in C_0^\infty(\widetilde{M})$ satisfies

$$\phi_k = \begin{cases} 1, & y \in B_{a_k}(x) \\ 0, & y \notin B_{a_{k-1}}(x), \end{cases}$$

$$0 \leq \phi_k \leq 1, \quad |\nabla_b \phi_k| \leq 2 \cdot 3^k.$$

Then iteratively we get from (3.14) that

$$\begin{aligned}
 (3.15) \quad & \left(\int_{B_{a_k}(x)} G_p^{q_{k+1}} dV_\theta \right)^{1/q_{k+1}} \\
 & \leq (Cq_k^2)^{1/q_k} (4 \cdot 3^{2k} + 1)^{1/q_k} \left(\int_{B_{a_{k-1}}(x)} G_p^{q_k} dV_\theta \right)^{1/q_k} \leq \dots \\
 & \leq \prod_{j=1}^k (Cr^{2j})^{1/pr^j} \left(\int_{B_{a_0}(x)} G_p^{b_n} dV_\theta \right)^{1/b_n}
 \end{aligned}$$

Since the product converges as $k \rightarrow \infty$, we can take the limit $k \rightarrow \infty$ in (3.15) to get

$$\sup_{y \in B_{\frac{1}{2}}(x)} G_p(y) \leq C \left(\int_{B_1(x)} G_p^{b_n} dV \right)^{1/b_n}.$$

This completes the proof. \square

4. INJECTIVITY OF THE CR DEVELOPING MAP

Let $p \in \widetilde{M}$, and recall that G_p denotes the minimal positive Green's function for the CR invariant sublaplacian D_θ with pole at p . Let $\Phi : \widetilde{M} \rightarrow S^{2n+1}$ be a CR developing map. Recall that H_y denotes the Green's function for the CR invariant sublaplacian D_0 of S^{2n+1} with the pole $y = \Phi(p)$. The normalized pullback of H_y is

$$\overline{G} := |\Phi'(p)|^{-(n+2)} |\Phi'|^n H_y \circ \Phi$$

which has poles in $\Phi^{-1}(y)$. Observe that Φ is one to one if and only if $\Phi^{-1}(y) = \{p\}$. Therefore to prove injectivity of Φ , it suffices to prove $G_p = \overline{G}$.

For any $y \in S^{2n+1}$, the Cayley transform is a global CR diffeomorphism

$$C_y : (S^{2n+1} \setminus \{y\}, \theta_{S^{2n+1}}) \rightarrow (H^n, \Theta)$$

with $C_y(y) = \infty$ and

$$C_y^* \Theta = H_y^{\frac{2}{n}} \theta_{S^{2n+1}}.$$

This means that $(S^{2n+1} \setminus \{y\}, H_y^{\frac{2}{n}} \theta_{S^{2n+1}})$ is Heisenberg flat. Now we have

$$\begin{aligned}
(4.1) |\Phi'(p)|^{\frac{-2(n+2)}{n}} \Phi^* \circ C_y^*(\Theta) &= |\Phi'(p)|^{\frac{-2(n+2)}{n}} \Phi^*(H_y^{\frac{2}{n}} \theta_{S^{2n+1}}) \\
&= |\Phi'(p)|^{\frac{-2(n+2)}{n}} (H_y \circ \Phi)^{\frac{2}{n}} \Phi^* \theta_{S^{2n+1}} \\
&= |\Phi'(p)|^{\frac{-2(n+2)}{n}} (H_y \circ \Phi)^{\frac{2}{n}} |\Phi'|^2 \theta \\
&= (|\Phi'(p)|^{-(n+2)} (H_y \circ \Phi) |\Phi'|^n)^{\frac{2}{n}} \theta \\
&= \overline{G}^{\frac{2}{n}} \theta.
\end{aligned}$$

It follows from, (4.1) that $(\widetilde{M}, \bar{\theta} := \overline{G}^{\frac{2}{n}} \theta)$ is Heisenberg flat away from $\Phi^{-1}(y)$.

Consider the quotient of G_p and \overline{G} :

$$v := \frac{G_p}{\overline{G}}.$$

By (3.5) we have $\overline{G} - G_p \geq 0$ away from $\Phi^{-1}(y)$. So there holds

$$0 < v \leq 1$$

away from $\Phi^{-1}(y)$. Taking $u = G_p$ and $\varphi = 1$ in (2.8), we obtain that on $\widetilde{M} \setminus \{p\}$:

$$(4.2) \quad R(G_p^{\frac{2}{n}} \theta) = G_p^{-1-\frac{2}{n}} D_\theta(G_p) = 0.$$

Here for a contact form η , $R(\eta)$ or R_η means the Tanaka-Webster scalar curvature with respect to η . Writing $G_p^{\frac{2}{n}} \theta = v^{\frac{2}{n}} \bar{\theta}$, we get

$$0 = R(G_p^{\frac{2}{n}} \theta) = R(v^{\frac{2}{n}} \bar{\theta}) = v^{-1-\frac{2}{n}} D_{\bar{\theta}}(v)$$

away from $\Phi^{-1}(y)$ by (4.2) and (2.8) with $u = v$ and $\varphi = 1$. Therefore we have

$$(4.3) \quad (b_n \bar{\Delta}_b + R_{\bar{\theta}})(v) = b_n \bar{\Delta}_b(v) = 0$$

by noting that $R_{\bar{\theta}} = 0$. We would like to examine the asymptotic behavior of v near $\Phi^{-1}(y)$. We will often write the coordinates (z_1, \dots, z_n, t) in H^n by (z, t) where $z = (z_1, \dots, z_n)$. Define the Heisenberg norm $\rho_0(z, t) := (|z|^4 + t^2)^{1/4}$. Denote the fundamental solution to $D_\Theta = b_n \Delta_b$ by $c(n) \rho_0(z, t)^{-2n}$ for some dimensional constant $c(n)$ ([12]).

Lemma 4.1. *For each $\bar{p} \in \Phi^{-1}(y)$, we can choose a coordinate map $(z, t) : \widetilde{M} \rightarrow H^n$ and a smooth function u near \bar{p} such that $(z(\bar{p}), t(\bar{p})) = (0, 0)$ and there hold*

$$(4.4) \quad G_p(z, t) = c(n) u(p) u \rho_0(z, t)^{-2n} + \text{a smooth function}$$

near $\bar{p} = p$ and

$$(4.5) \quad \bar{G}(z, t) = c(n)u(q)u\rho_0(z, t)^{-2n} + a \text{ smooth function}$$

near $\bar{p} = q$, resp..

Proof. Let $-y \in S^{2n+1}$ be the antipodal point of $y \in S^{2n+1}$. Consider the Cayley transform C_{-y} (with pole at $-y$). Obviously, $C_{-y}(y) = 0 \in H^n$ and

$$(4.6) \quad (C_{-y}^{-1})^*(H_{-y}^{\frac{2}{n}}\theta_{S^{2n+1}}) = \Theta$$

It follows from (4.6) that

$$(4.7) \quad (C_{-y} \circ \Phi)^*\Theta = (H_{-y} \circ \Phi)^{\frac{2}{n}}|\Phi'|^2\theta.$$

Here we have written $\Phi^*(\theta_{S^{2n+1}}) = |\Phi'|^2\theta$. Let $u := (H_{-y} \circ \Phi)|\Phi'|^n$. We can then write (4.7) as

$$(4.8) \quad (C_{-y} \circ \Phi)^*\Theta = u^{\frac{2}{n}}\theta$$

near $q \in \Phi^{-1}(y)$. Take $C_{-y} \circ \Phi : \widetilde{M} \rightarrow H^n$ as a coordinate map (z, t) . By (4.8) and (2.8) in Section 2 with $\varphi = c(n)\rho_0(z, t)^{-2n}$, we obtain

$$\begin{aligned} D_\theta(uc(n)\rho_0(z, t)^{-2n}) &= u^{1+\frac{2}{n}}D_\Theta(c(n)\rho_0(z, t)^{-2n}) \\ &= u^{1+\frac{2}{n}}\delta_{(0,0)}. \end{aligned}$$

Note that the volume change formula is $((C_{-y} \circ \Phi)^*) \Theta \wedge (d\Theta)^n = u^{2+\frac{2}{n}}\theta \wedge (d\theta)^n$. So in view of Theorem 3.4, (3.3), and (3.4), we have

$$(4.9) \quad \begin{aligned} D_\theta(G_p - u(p)uc(n)\rho_0(z, t)^{-2n}) &= 0, \\ D_\theta(\bar{G} - u(\bar{p})uc(n)\rho_0(z, t)^{-2n}) &= 0 \end{aligned}$$

near $p, \bar{p} \in \Phi^{-1}(y)$, resp.. By applying (2.8) we obtain

$$(4.10) \quad \begin{aligned} D_{(C_{-y} \circ \Phi)^*\Theta}(u^{-1}(G_p - u(p)uc(n)\rho_0(z, t)^{-2n})) &= 0 \\ D_{(C_{-y} \circ \Phi)^*\Theta}(u^{-1}(\bar{G} - u(\bar{p})uc(n)\rho_0(z, t)^{-2n})) &= 0 \end{aligned}$$

according to (4.8) and (4.9). Observe that $R_{(C_{-y} \circ \Phi)^*\Theta} = (C_{-y} \circ \Phi)^*R_\Theta = R_\Theta \circ (C_{-y} \circ \Phi) = 0$. Hence $D_{(C_{-y} \circ \Phi)^*\Theta} = b_n\Delta_b$. By the regularity result for Δ_b (e.g., Theorem 16.7 in [12]), we have (4.4) and (4.5) from (4.10) and $G_p, \bar{G}, \rho_0(z, t)^{-2n}$ being L_{loc}^1 .

□

By Lemma 4.1 we deduce that near p (which is a pole of \bar{G}):

$$v(z, t) = 1 + O(\rho_0^{2n}),$$

and near $\bar{p} \in \Phi^{-1}(y) \setminus \{p\}$

$$v(z, t) = O(\rho_0^{2n}).$$

since G_p is smooth near any $\bar{p} \in \Phi^{-1}(y) \setminus \{p\}$. From Lemma 4.1 we also have

$$\begin{aligned} |\nabla_b v| &= O(\rho_0^{2n-1}), \\ |\nabla_b |\nabla_b v|| &= O(\rho_0^{2n-2}), \Delta_b v = O(\rho_0^{2n-2}) \end{aligned}$$

near any $\bar{p} \in \Phi^{-1}(y)$.

With the asymptotic behavior of G_p , \bar{G} , and v near any $\bar{p} \in \Phi^{-1}(y)$ understood, we observe that the set $\Phi^{-1}(y)$ has no contribution when we play the integration by parts in the computation below throughout the remaining section. We would like to show that v is a constant, and hence is identically one since $v(p) = 1$. Write G , dV instead of G_p , dV_θ for short in the remaining section. Also note that the notation C may mean different constants.

Lemma 4.2. *There exists a constant $C > 0$ such that for any $\phi \in C_0^\infty(\widetilde{M})$ there holds*

$$(4.11) \quad \int_{\widetilde{M}} \phi^2 |\nabla_b \log \bar{G}|^2 dV \leq C \int_{\widetilde{M}} (\phi^2 |\nabla_b \log G|^2 + |\nabla_b \phi|^2) dV.$$

Proof. Away from $\Phi^{-1}(y)$ we have

$$\begin{aligned} \Delta_b \log \bar{G} &= \bar{G}^{-1} \Delta_b \bar{G} - (-\bar{G}^{-2} |\nabla_b \bar{G}|^2) \\ &= \bar{G}^{-1} \Delta_b \bar{G} + |\nabla_b \log \bar{G}|^2 \\ (4.12) \quad &= -b_n^{-1} R(\theta) + |\nabla_b \log \bar{G}|^2 \\ &= G^{-1} \Delta_b G + |\nabla_b \log \bar{G}|^2 \\ &= \Delta_b \log G - |\nabla_b \log G|^2 + |\nabla_b \log \bar{G}|^2 \end{aligned}$$

Multiplying ϕ^2 on both sides of (4.12) and integrating by parts, we obtain

$$\begin{aligned} \int_{\widetilde{M}} \phi^2 |\nabla_b \log \bar{G}|^2 dV &= \int_{\widetilde{M}} \phi^2 |\nabla_b \log G|^2 dV + 2 \int_{\widetilde{M}} \phi \nabla_b \phi (\nabla_b \log \bar{G} - \nabla_b \log G) dV \\ &\leq C \int_{\widetilde{M}} (\phi^2 |\nabla_b \log G|^2 + |\nabla_b \phi|^2) dV. \end{aligned}$$

for some constant C by noting that the boundary integral around $\bar{p} \in \Phi^{-1}(y)$ tends to zero. Here we have used the Schwarz inequality with ε in the last inequality.

□

Let \bar{T} ($\bar{\nabla}_b, \bar{\Delta}_b, d\bar{V}, \bar{R} = R_{\bar{\theta}}$, etc., resp.) denote the corresponding quantity of T ($\nabla_b, \Delta_b, dV, R$, etc., resp.) with respect to $\bar{\theta}$ (while fixing J).

Lemma 4.3. *There holds $v_{\bar{0}} := \bar{T}v \equiv 0$ in either of the following cases:*

- (a) $n \geq 3$
- (b) $n = 2$ and $s(M) < 1$.

Proof. First observe that the Paneitz-like operator P is nonnegative for $\phi \in C_0^\infty(\widetilde{M})$ in 2.4 if $n \geq 2$ (Extending Theorem 3.2 in [5] to this situation). With respect to $\bar{\theta}$ (Heisenberg flat), the torsion vanishes and hence $\kappa = 0$ in (2.6). Therefore by extending (2.6) to the situation for $\bar{\theta}$ (singular at $\bar{p} \in \Phi^{-1}(y)$) in view of the asymptotic behavior of v discussed before, we have

$$\begin{aligned}
 n^2 \int_{\widetilde{M}} (\phi v)_0^2 d\bar{V} &\leq \int_{\widetilde{M}} (\bar{\Delta}_b(\phi v))^2 d\bar{V} \\
 (4.13) \quad &= \int_{\widetilde{M}} (\phi \bar{\Delta}_b v + v \bar{\Delta}_b \phi - 2 \bar{\nabla}_b \phi \bar{\nabla}_b v)^2 d\bar{V} \\
 &\leq C \left(\int_{\widetilde{M}} v^2 (\bar{\Delta}_b \phi)^2 d\bar{V} + \int_{\widetilde{M}} |\bar{\nabla}_b \phi|^2 |\bar{\nabla}_b v|^2 d\bar{V} \right) \\
 &= C(I + II)
 \end{aligned}$$

for $\phi \in C_0^\infty(\widetilde{M})$, where $I = \int_{\widetilde{M}} v^2 (\bar{\Delta}_b \phi)^2 d\bar{V}$ and $II = \int_{\widetilde{M}} |\bar{\nabla}_b \phi|^2 |\bar{\nabla}_b v|^2 d\bar{V}$. Rewrite

$$\begin{aligned}
 (4.14) \quad II &= \int_{\widetilde{M}} |\bar{\nabla}_b \phi|^2 |\bar{\nabla}_b v|^2 d\bar{V} \\
 &= \int_{\widetilde{M}} |\nabla_b \phi|^2 |\nabla_b v|^2 \bar{G}^{\frac{2n-2}{n}} dV
 \end{aligned}$$

where

$$\begin{aligned}
 (4.15) \quad |\nabla_b v|^2 &= \left| \frac{\bar{G} \nabla_b G - G \nabla_b \bar{G}}{\bar{G}^2} \right|^2 \\
 &= |v \nabla_b \log G - v \nabla_b \log \bar{G}|^2 \\
 &\leq C(v^2 |\nabla_b \log G|^2 + v^2 |\nabla_b \log \bar{G}|^2).
 \end{aligned}$$

Let $q' := \frac{2(n-1)}{n}$. Then $q' > 1$ if and only if $n \geq 3$. From (4.15) we have

$$\begin{aligned}
 \overline{G}^{q'} |\nabla_b v|^2 &\leq C(v^2 \overline{G}^{q'} |\nabla_b \log G|^2 + v^2 \overline{G}^{q'} |\nabla_b \log \overline{G}|^2) \\
 (4.16) \quad &= C v^{2-q'} (G^{q'} |\nabla_b \log G|^2 + G^{q'} |\nabla_b \log \overline{G}|^2) \\
 &\leq C G^{q'} (|\nabla_b \log G|^2 + |\nabla_b \log \overline{G}|^2).
 \end{aligned}$$

Recall that $B_\rho(p) \subset \widetilde{M}$ denote the ball of radius ρ , centered at p , with respect to the Carnot-Carathéodory distance. Substituting (4.16) into (4.14), we get

$$(4.17) \quad II \leq C \rho^{-2} \int_{B_{2\rho}(p) \setminus B_\rho(p)} G^{q'} (|\nabla_b \log G|^2 + |\nabla_b \log \overline{G}|^2) dV$$

by taking a cutoff function ϕ such that $0 \leq \phi \leq 1$, $\phi = 1$ on $B_\rho(p)$, $\phi = 0$ on $(\widetilde{M} \setminus B_{2\rho}(p))$, and $|\nabla_b \phi| \leq \frac{C}{\rho}$.

Taking $\phi = \psi G^{\frac{q'}{2}}$ in (4.11), $\psi \in C_0^\infty(\widetilde{M} \setminus \{p\})$, we get

$$\begin{aligned}
 (4.18) \quad &\int_{\widetilde{M}} \psi^2 G^{q'} |\nabla_b \log \overline{G}|^2 dV \\
 &\leq C \int_{\widetilde{M}} (\psi^2 G^{q'} |\nabla_b \log G|^2 + |\nabla_b (G^{q'/2} \psi)|^2) dV \\
 &\leq C \int_{\widetilde{M}} (\psi^2 |\nabla_b \log G|^2 + |\nabla_b \psi|^2) G^{q'} dV.
 \end{aligned}$$

Note that the integral of $|\nabla_b \log G|^2 G^{q'}$ over a region containing p diverges (this is why we need ψ compactly supported away from p). Choosing ψ such that $0 \leq \psi \leq 1$, $\psi = 1$ on $B_{2\rho}(p) \setminus B_\rho(p)$, $\psi = 0$ on $B_{\rho/2}(p) \cup (\widetilde{M} \setminus B_{4\rho}(p))$, and $|\nabla_b \psi| \leq \frac{C}{\rho}$, we get

$$(4.19) \quad II \leq C \rho^{-2} \int_{B_{4\rho}(p) \setminus B_{\rho/2}(p)} G^{q'} (1 + |\nabla_b \log G|^2) dV$$

from (4.17) and (4.18) (for ρ large).

For $I = \int_{\widetilde{M}} v^2 (\overline{\Delta}_b \phi)^2 d\overline{V}$, since

$$(b_n \overline{\Delta}_b + \overline{R}) \phi = \overline{G}^{-1-\frac{2}{n}} (b_n \Delta_b + R_\theta) (\overline{G} \phi),$$

we have

$$\begin{aligned}
 (4.20) \quad &b_n \overline{\Delta}_b \phi = \overline{G}^{-1-\frac{2}{n}} (b_n (\phi \Delta_b \overline{G} + \overline{G} \Delta_b \phi - 2 \nabla_b \phi \nabla_b \overline{G}) + R_\theta \overline{G} \phi) \\
 &= \overline{G}^{-1-\frac{2}{n}} (\phi (b_n \Delta_b \overline{G} + R_\theta \overline{G}) + b_n (\overline{G} \Delta_b \phi - 2 \nabla_b \phi \nabla_b \overline{G})) \\
 &= b_n (\overline{G}^{-\frac{2}{n}} \Delta_b \phi - 2 \overline{G}^{-1-\frac{2}{n}} \nabla_b \phi \nabla_b \overline{G}),
 \end{aligned}$$

that is,

$$(4.21) \quad \overline{\Delta}_b \phi = \overline{G}^{-\frac{2}{n}} \Delta_b \phi - 2\overline{G}^{-1-\frac{2}{n}} \nabla_b \phi \nabla_b \overline{G}.$$

By (4.21) we have

$$(4.22) \quad \begin{aligned} I &= \int_{\widetilde{M}} v^2 (\overline{G}^{-\frac{2}{n}} \Delta_b \phi - 2\overline{G}^{-1-\frac{2}{n}} \nabla_b \phi \nabla_b \overline{G})^2 d\overline{V} \\ &\leq C \left(\int_{\widetilde{M}} v^2 \left(\frac{\Delta_b \phi}{\overline{G}^{\frac{2}{n}}} \right)^2 d\overline{V} + \int_{\widetilde{M}} v^2 \overline{G}^{-2-\frac{4}{n}} |\nabla_b \phi|^2 |\nabla_b \overline{G}|^2 d\overline{V} \right) \\ &= C(III + IV) \end{aligned}$$

where

$$(4.23) \quad \begin{aligned} III &= \int_{\widetilde{M}} v^2 \left(\frac{\Delta_b \phi}{\overline{G}^{\frac{2}{n}}} \right)^2 d\overline{V} \\ &= \int_{\widetilde{M}} \left(\frac{\Delta_b \phi}{\overline{G}^{\frac{2}{n}}} \right)^2 \left(\frac{G}{\overline{G}} \right)^2 (\overline{G}^{\frac{2}{n}})^{n+1} dV \\ &= \int_{\widetilde{M}} (\Delta_b \phi)^2 G^2 \overline{G}^{\frac{2(n+1)}{n} - \frac{4}{n} - 2} dV \\ &\leq C \rho^{-2} \int_{\widetilde{M} \setminus B_{\rho/4}} G^{2-\frac{2}{n}} dV \end{aligned}$$

and

$$(4.24) \quad \begin{aligned} IV &= \int_{\widetilde{M}} v^2 \overline{G}^{-2-\frac{4}{n}} |\nabla_b \phi|^2 |\nabla_b \overline{G}|^2 dV \\ &= \int_{\widetilde{M}} v^2 \overline{G}^{-\frac{2}{n}} |\nabla_b \phi|^2 |\nabla_b \overline{G}|^2 dV \\ &= \int_{\widetilde{M}} v^2 \overline{G}^{2-\frac{2}{n}} |\nabla_b \phi|^2 |\nabla_b \log \overline{G}|^2 dV \\ &\leq \int_{\widetilde{M}} |\nabla_b \phi|^2 G^{q'} |\nabla_b \log \overline{G}|^2 dV \\ &\leq C \rho^{-2} \int_{B_{2\rho}(p) \setminus B_{\rho}(p)} G^{q'} |\nabla_b \log \overline{G}|^2 dV \end{aligned}$$

by a suitable choice of ϕ . Again we let $\phi = \psi G^{q'/2}$ in Lemma 4.2, where $\psi \in C_0^\infty(\widetilde{M} \setminus \{p\})$. By choosing ψ suitably, we can convert (4.24) into

$$(4.25) \quad IV \leq C \rho^{-2} \int_{B_{4\rho}(p) \setminus B_{\rho/2}(p)} G^{q'} (1 + |\nabla_b \log G|^2) dV.$$

Finally we are going to show that

$$(4.26) \quad II \text{ (IV, resp.)} \leq C\rho^{-2} \int_{\widetilde{M} \setminus B_{\rho/4}} G^{q'} dV$$

(recall that $q' = 2 - \frac{2}{n}$) for $n \geq 3$, and

$$(4.27) \quad II \text{ (IV, resp.)} \leq C\rho^{-2} \int_{\widetilde{M} \setminus B_{\rho/4}} G^{\tilde{q}} dV$$

for $n = 2$, in which $\tilde{q} < 1$. First consider the case (a) $n \geq 3$. So $q' > 1$. Now $b_n \Delta_b G + R_\theta G = 0$ (away from p) and $R_\theta > C > 0$. This implies that

$$\Delta_b G = -b_n^{-1} R_\theta G \leq -b_n^{-1} C G$$

Multiplying by $\phi^2 G^{q'-1}$ with $\phi \in C_0^\infty(\widetilde{M} \setminus \{p\})$ and integrating by parts give

$$\begin{aligned} 0 &\geq \int_{\widetilde{M}} \phi^2 G^{q'-1} \Delta_b G dV \\ &= \int_{\widetilde{M}} \nabla_b(\phi^2 G^{q'-1}) \nabla_b G dV \\ &= \int_{\widetilde{M}} 2\phi G^{q'-1} \nabla_b \phi \nabla_b G dV + (q' - 1) \int_{\widetilde{M}} \phi^2 G^{q'-2} |\nabla_b G|^2 dV. \end{aligned}$$

By Young's inequality with ε , we get

$$(4.28) \quad \int_{\widetilde{M}} \phi^2 G^{q'-2} |\nabla_b G|^2 dV \leq C \int_{\widetilde{M}} G^{q'} |\nabla_b \phi|^2 dV$$

(noting that we have used the fact $q' > 1$). By choosing ϕ appropriately in (4.28) and observing that $G^{q'-2} |\nabla_b G|^2 = G^{q'} |\nabla_b \log G|^2$, we obtain

$$(4.29) \quad \int_{B_{4\rho}(p) \setminus B_{\rho/2}(p)} G^{q'} |\nabla_b \log G|^2 \leq C\rho^{-2} \int_{B_{8\rho}(p) \setminus B_{\rho/4}(p)} G^{q'} dV.$$

So we have (4.26) for II (IV, resp.) by (4.19) and (4.29) ((4.25) and (4.29), resp.).

For the case (b), $n = 2$ implies $q' = 1$. From $b_n \Delta_b G + R_\theta G = 0$ where $|R_\theta| \leq c$ we have

$$\Delta_b G = -b_n^{-1} R_\theta G \geq -b_n^{-1} c G,$$

that is,

$$0 \leq \Delta_b G + b_n^{-1} c G$$

Multiplying by $\phi^2 G^{\tilde{q}-1}$ with $\tilde{q} < 1$, $\phi \in C_0^\infty(M \setminus \{p\})$ and integrating by parts give

$$\begin{aligned}
(4.30) \quad 0 &\leq \int_{\tilde{M}} \phi^2 G^{\tilde{q}-1} \Delta_b G dV + b_n^{-1} c \int_{\tilde{M}} \phi^2 G^{\tilde{q}} dV \\
&= \int_{\tilde{M}} \nabla_b (\phi^2 G^{\tilde{q}-1}) \nabla_b G dV + b_n^{-1} c \int_{\tilde{M}} \phi^2 G^{\tilde{q}} dV \\
&= \int_{\tilde{M}} 2\phi G^{\tilde{q}-1} \nabla_b \phi \nabla_b G dV + b_n^{-1} c \int_{\tilde{M}} \phi^2 G^{\tilde{q}} dV + (\tilde{q} - 1) \int_{\tilde{M}} \phi^2 G^{\tilde{q}-2} |\nabla_b G|^2 dV.
\end{aligned}$$

From (4.30) we have

$$\begin{aligned}
&(1 - \tilde{q}) \int_{\tilde{M}} \phi^2 G^{\tilde{q}-2} |\nabla_b G|^2 dV \\
&\leq 2 \int_{\tilde{M}} \phi G^{\tilde{q}-1} |\nabla_b \phi| |\nabla_b G| dV + b_n^{-1} c \int_{\tilde{M}} \phi^2 G^{\tilde{q}} dV
\end{aligned}$$

By Young's inequality with ε , we obtain

$$\int_{\tilde{M}} \phi^2 G^{\tilde{q}-2} |\nabla_b G|^2 dV \leq C \left(\int_{\tilde{M}} G^{\tilde{q}} |\nabla_b \phi|^2 dV + \int_{\tilde{M}} G^{\tilde{q}} \phi^2 dV \right).$$

Since $G^{\tilde{q}-2} \geq G^{-1}$ on $\tilde{M} \setminus K$ for some compact subset K by Proposition 3.6, we have

$$(4.31) \quad \int_{\tilde{M}} \phi^2 G^{-1} |\nabla_b G|^2 dV \leq C \left(\int_{\tilde{M}} G^{\tilde{q}} |\nabla_b \phi|^2 dV + \int_{\tilde{M}} G^{\tilde{q}} \phi^2 dV \right).$$

Observing that $G^{-1} |\nabla_b G|^2 = G |\nabla_b \log G|^2$ and choosing a suitable cut-off function ϕ in (4.31), we obtain

$$(4.32) \quad \int_{B_{4\rho}(p) \setminus B_{\rho/2}(p)} G |\nabla_b \log G|^2 dV \leq C \left(\frac{1}{\rho^2} + 1 \right) \int_{B_{8\rho}(p) \setminus B_{\rho/4}(p)} G^{\tilde{q}} dV.$$

Thus we have (4.27) for II (IV , resp.) by (4.19) and (4.32) ((4.25) and (4.32), resp.) for ρ large in view of Proposition 3.6.

By (3.7) and the assumption $s(M) < 1$ for $n = 2$, we have the convergence of the integrals in (4.26) and (4.27) (in fact, both of them tend to zero as $\rho \rightarrow \infty$). So as $\rho \rightarrow \infty$, II and IV go to zero. On the other hand, it is clear that III goes to zero as $\rho \rightarrow \infty$ by (4.23) and (3.7). So from (4.13) and (4.22) we conclude that $v_0 = 0$.

□

Proof. (of Theorem A)

We need only to prove $v := \frac{G}{\bar{G}} \equiv 1$. Let $q = \frac{2n}{n+1}$. We first prove that for any $\phi \in C_0^\infty(\widetilde{M})$ there holds

$$(4.33) \quad \int_{\widetilde{M}} \phi^2 |\bar{\nabla}_b v|^{q-2} |\bar{\nabla}_b |\bar{\nabla}_b v||^2 d\bar{V} \leq C \int_{\widetilde{M}} |\nabla_b \phi|^2 \bar{G}^q |\nabla_b v|^q dV$$

for some constant C . Since $\bar{\Delta}_b v = 0$, $\bar{\theta} = \bar{G}^{\frac{2}{n}} \theta$ is flat (hence Ric and Tor vanish), and $v_{\bar{0}} = 0$ by Lemma 4.3, we reduce the Bochner formula (2.2) to

$$(4.34) \quad \frac{1}{2} \bar{\Delta}_b |\bar{\nabla}_b v|^2 = -|(\bar{\nabla}^H)^2 v|^2.$$

Observe that

$$(4.35) \quad |\bar{\nabla}_b |\bar{\nabla}_b v||^2 \leq |(\bar{\nabla}^H)^2 v|^2.$$

For $q > 1$ we compute

$$(4.36) \quad \begin{aligned} \bar{\Delta}_b |\bar{\nabla}_b v|^q &= \bar{\Delta}_b (|\bar{\nabla}_b v|^2)^{\frac{q}{2}} \\ &= \frac{q}{2} (|\bar{\nabla}_b v|^2)^{\frac{q}{2}-1} \bar{\Delta}_b |\bar{\nabla}_b v|^2 - \frac{q}{2} \left(\frac{q}{2} - 1\right) (|\bar{\nabla}_b v|^2)^{\frac{q}{2}-2} |\bar{\nabla}_b |\bar{\nabla}_b v||^2 \\ &= \frac{q}{2} |\bar{\nabla}_b v|^{q-2} \bar{\Delta}_b |\bar{\nabla}_b v|^2 - \frac{q(q-2)}{4} |\bar{\nabla}_b v|^{q-4} |\bar{\nabla}_b |\bar{\nabla}_b v||^2 \\ &\leq -q(q-1) |\bar{\nabla}_b v|^{q-2} |\bar{\nabla}_b |\bar{\nabla}_b v||^2 \\ &= -C_q |\bar{\nabla}_b v|^{q-2} |\bar{\nabla}_b |\bar{\nabla}_b v||^2 \end{aligned}$$

where $C_q = q(q-1) > 0$ for $n \geq 2$. For the inequality in (4.36) we have used (4.34), (4.35). Consider first the case $\phi \in C_0^\infty(\widetilde{M} \setminus \Phi^{-1}(y))$. Multiplying (4.36) by ϕ^2 and integrating by parts, we get

$$\begin{aligned} \int_{\widetilde{M}} \phi^2 |\bar{\nabla}_b |\bar{\nabla}_b v||^2 |\bar{\nabla}_b v|^{q-2} d\bar{V} &\leq -C_q^{-1} \int_{\widetilde{M}} \phi^2 \bar{\Delta}_b |\bar{\nabla}_b v|^q d\bar{V} \\ &= -C_q^{-1} \int_{\widetilde{M}} \bar{\nabla}_b (\phi^2) \bar{\nabla}_b |\bar{\nabla}_b v|^q d\bar{V} \\ &\leq 2q C_q^{-1} \int_{\widetilde{M}} |\phi| |\bar{\nabla}_b \phi| |\bar{\nabla}_b v|^{q-1} |\bar{\nabla}_b |\bar{\nabla}_b v|| d\bar{V} \\ &\leq C'_q \int_{\widetilde{M}} |\bar{\nabla}_b \phi|^2 |\bar{\nabla}_b v|^q d\bar{V} \end{aligned}$$

for some constant C'_q , where the last inequality is deduced by applying the Schwarz (or Young's) inequality with ε . Now we change the integral

on the right hand side to a corresponding one using the form θ and get

$$\begin{aligned} \int_{\widetilde{M}} \phi^2 |\overline{\nabla}_b |\overline{\nabla}_b v||^2 |\overline{\nabla}_b v|^{q-2} d\overline{V} &\leq C'_q \int_{\widetilde{M}} \overline{G}^{-\frac{2}{n}} |\nabla_b \phi|^2 |\nabla_b v|^q \overline{G}^{-\frac{q}{n}} (\overline{G}^{\frac{2}{n}})^{n+1} dV \\ &= C'_q \int_{\widetilde{M}} |\nabla_b \phi|^2 |\nabla_b v|^q \overline{G}^{\frac{2(n+1)-2-q}{n}} dV. \end{aligned}$$

Observe that $q = \frac{2n}{n+1}$ implies $\frac{2(n+1)-2-q}{n} = q$. We have shown the desired inequality (4.33) for $\phi \in C_0^\infty(\widetilde{M} \setminus \Phi^{-1}(y))$.

Now for $\phi \in C_0^\infty(\widetilde{M})$, we consider $\psi_r \phi$ where ψ_r is a cutoff function such that for each $\bar{p} \in \Phi^{-1}(y)$, $\psi_r \equiv 0$ in $B_r(\bar{p})$, $\psi_r \equiv 1$ on $\widetilde{M} \setminus B_{2r}(\bar{p})$, and $0 \leq \psi_r \leq 1$ (r small so that $B_{2r}(\bar{p}_1) \cap B_{2r}(\bar{p}_2)$ is empty for any pair of points $\bar{p}_1, \bar{p}_2 \in \Phi^{-1}(y)$). We also require that $|\nabla_b \psi_r| \leq 2r^{-1}$. Applying (4.33) for $\psi_r \phi$, we have

$$\begin{aligned} &\int_{\widetilde{M}} \psi_r^2 \phi^2 |\overline{\nabla}_b |\overline{\nabla}_b v||^2 |\overline{\nabla}_b v|^{q-2} d\overline{V} \\ &\leq C_1 \int_{\widetilde{M}} \psi_r^2 |\nabla_b \phi|^2 |\nabla_b v|^q \overline{G}^q dV + C_2 \int_{\widetilde{M}} \phi^2 |\nabla_b \psi_r|^2 |\nabla_b v|^q \overline{G}^q dV. \end{aligned}$$

Noticing that the last integral has order $O(r^{2n-q}) \rightarrow 0$, as $r \rightarrow 0$, we see that (4.33) holds for $\phi \in C_0^\infty(\widetilde{M})$.

Next we are going to prove

$$\begin{aligned} (4.37) \quad &\int_{B_\rho(p)} |\overline{\nabla}_b v|^{q-2} |\overline{\nabla}_b |\overline{\nabla}_b v||^2 d\overline{V} \\ &\leq C\rho^{-2} \int_{B_{4\rho}(p) \setminus B_{\rho/2}(p)} G^q (1 + |\nabla_b \log G|^2) dV, \end{aligned}$$

where $\rho > 0$ is sufficiently large and C is a constant. We want to make use of (4.33). Note that

$$\begin{aligned} (4.38) \quad &\overline{G}^q |\nabla_b v|^q = |\overline{G} \nabla_b v|^q \\ &= |\nabla_b G - G \overline{G}^{-1} \nabla_b \overline{G}|^q \\ &\leq C(|\nabla_b G|^q + G^q |\nabla_b \log \overline{G}|^q). \end{aligned}$$

Thus, if we take $\phi \in C_0^\infty(\widetilde{M})$ such that $\phi \equiv 1$ in $B_\rho(p)$, $\phi \equiv 0$ on $\widetilde{M} \setminus B_{2\rho}(p)$, $0 \leq \phi \leq 1$ and $|\nabla_b \phi| \leq 2\rho^{-1}$, We see from (4.33) and (4.38) that

(4.39)

$$\begin{aligned} \int_{B_\rho(p)} |\bar{\nabla}_b v|^{q-2} |\bar{\nabla}_b |\bar{\nabla}_b v||^2 d\bar{V} &\leq \int_{\widetilde{M}} \phi^2 |\bar{\nabla}_b v|^{q-2} |\bar{\nabla}_b |\bar{\nabla}_b v||^2 d\bar{V} \\ &\leq C \rho^{-2} \int_{B_{2\rho}(p) \setminus B_\rho(p)} (|\nabla_b G|^q + G^q |\nabla_b \log \bar{G}|^q) dV. \end{aligned}$$

Let $a = \frac{(2-q)q}{2}$. Note that $q < 2$. By Young's inequality we have

$$\begin{aligned} |\nabla_b G|^q &= G^a G^{-a} |\nabla_b G|^q \\ &\leq C(G^{a\frac{2}{2-q}} + G^{-a\frac{2}{q}} |\nabla_b G|^2) \\ &\leq C(G^q + G^{q-2} |\nabla_b G|^2) \\ &= CG^q(1 + |\nabla_b \log G|^2), \end{aligned} \tag{4.40}$$

and

$$\begin{aligned} G^q |\nabla_b \log \bar{G}|^q &= G^q \frac{|\nabla_b \bar{G}|^q}{\bar{G}^q} \\ &\leq G^q \frac{C \bar{G}^q (1 + |\nabla_b \log \bar{G}|^2)}{\bar{G}^q} \\ &= CG^q(1 + |\nabla_b \log \bar{G}|^2). \end{aligned} \tag{4.41}$$

So from (4.39), (4.40), and (4.41), we obtain

$$\begin{aligned} &\int_{B_\rho(p)} |\bar{\nabla}_b v|^{q-2} |\bar{\nabla}_b |\bar{\nabla}_b v||^2 d\bar{V} \\ &\leq C \rho^{-2} \int_{B_{2\rho}(p) \setminus B_\rho(p)} G^q (1 + |\nabla_b \log G|^2 + |\nabla_b \log \bar{G}|^2) dV. \end{aligned} \tag{4.42}$$

Taking $\phi = \psi G^{\frac{q}{2}}$ in Lemma 4.2, where $\psi \in C_0^\infty(\widetilde{M} \setminus \{p\})$, we have

$$\begin{aligned} &\int_{\widetilde{M}} \psi^2 G^q |\nabla_b \log \bar{G}|^2 dV \\ &\leq C \int_{\widetilde{M}} (\psi^2 G^q |\nabla_b \log G|^2 + |\nabla_b (\psi G^{\frac{q}{2}})|^2) dV \\ &\leq C \int_{\widetilde{M}} (\psi^2 |\nabla_b \log G|^2 + |\nabla_b \psi|^2) G^q dV. \end{aligned} \tag{4.43}$$

Choosing the cutoff function ψ appropriately in (4.43), we get

$$(4.44) \quad \begin{aligned} & \int_{B_{2\rho} \setminus B_\rho} G^q |\nabla_b \log \bar{G}|^2 dV \\ & \leq C \int_{B_{4\rho}(p) \setminus B_{\rho/2}(p)} G^q (1 + |\nabla_b \log G|^2) dV \end{aligned}$$

for ρ large. Substituting (4.44) into (4.42) gives (4.37).

Since $b_n \Delta_b G + R_\theta G = 0$ on $\widetilde{M} \setminus \{p\}$ and $R_\theta \geq C > 0$, we have

$$\Delta_b G = -b_n^{-1} R_\theta G \leq -b_n^{-1} C G$$

Multiplying by $\phi^2 G^{q-1}$ with $\phi \in C_0^\infty(\widetilde{M} \setminus \{p\})$ and integrating by parts give

$$(4.45) \quad \begin{aligned} 0 & \geq \int_{\widetilde{M}} \phi^2 G^{q-1} (\Delta_b G) dV \\ & = \int_{\widetilde{M}} \nabla_b (\phi^2 G^{q-1}) \nabla_b G dV \\ & = \int_{\widetilde{M}} 2\phi G^{q-1} \nabla_b \phi \nabla_b G dV + (q-1) \int_{\widetilde{M}} \phi^2 G^{q-2} |\nabla_b G|^2 dV. \end{aligned}$$

Applying the Schwarz inequality with ε to (4.45), we obtain

$$(4.46) \quad \int_{\widetilde{M}} \phi^2 G^{q-2} |\nabla_b G|^2 dV \leq C \int_{\widetilde{M}} G^q |\nabla_b \phi|^2 dV.$$

Noting that $G^{q-2} |\nabla_b G|^2 = G^q |\nabla_b \log G|^2$ and choosing some appropriate ϕ in (4.46), we can reduce (4.37) to

$$(4.47) \quad \int_{B_\rho(p)} |\bar{\nabla}_b v|^{q-2} |\bar{\nabla}_b |\bar{\nabla}_b v||^2 dV \leq C \rho^{-2} \int_{\widetilde{M} \setminus B_{\rho/4}} G^q dV.$$

By (3.7) G^q is integrable since $q = \frac{2n}{n+1} > 1$ for $n \geq 2$. So letting $\rho \rightarrow \infty$ in (4.47) we get

$$|\bar{\nabla}_b v| = \text{const.}$$

Since $|\bar{\nabla}_b v| = \bar{G}^{-1} |\bar{\nabla}_b G - v \bar{\nabla}_b \bar{G}| \rightarrow 0$ at p by Lemma 4.1, we have $\bar{\nabla}_b v = 0$. So $v = \text{const.}$ From $v \rightarrow 1$ at p , we conclude that $v \equiv 1$. \square

5. THE POSITIVE CR MASS THEOREM

In this section, according to the work of Li in [23], we would like to introduce a positive mass theroem for spherical CR manifolds. Let M be a closed spherical CR manifold, \widetilde{M} be its universal cover and

$$\pi : \widetilde{M} \longrightarrow M$$

be the canonical projection map,

$$\Phi : \widetilde{M} \longrightarrow S^{2n+1}$$

be a CR developing map. We would like to construct local coordinates near each point b of M . There is a local inverse π^{-1} as follows:

$$\pi^{-1} : U_b \longrightarrow \widetilde{M},$$

where U_b is a neighborhood of $b \in M$. Let $q = \Phi(p) \in S^{2n+1}$, where $p \in \pi^{-1}(b)$, the local CR transformation

$$T = C_q \circ \Phi \circ \pi^{-1} : U_b \longrightarrow H^n$$

provides M a local coordinate $(z, t) \in H^n$ such that $(z(b), t(b)) = \infty$. Here $C_q : S^{2n+1} \longrightarrow H^n$ is the Cayley transform with pole at q , i.e. $C_q(q) = \infty$. We will call such coordinates "spherical CR coordinates near ∞ ".

Let G_b be the Green's function of D_θ with pole at b . It follows that there is a positive smooth function $h = h(z, t)$ defined on H^n near ∞ such that

$$(5.1) \quad (T^{-1})^*(G_b^{\frac{2}{n}} \theta) = h^{\frac{2}{n}} \Theta.$$

By positive constant rescaling we may assume that the complex Jacobian at p is $|\Phi'(p)| = 1$. Let $\rho(z, t) = (|z|^4 + t^2)^{1/4}$ be the Heisenberg norm on H^n . Therefore, We have the following asymptotic expansion of $h = h(z, t)$ near ∞ :

Lemma 5.1. *Let M be a closed spherical CR manifold which is not the standard sphere. Suppose the CR developing map is injective. Let h be defined as above. Then we have, near ∞ ,*

$$h = h(z, t) > 1$$

and

$$h(z, t) = 1 + A_b \cdot \rho(z, t)^{-2n} + O(\rho(z, t)^{-2n-1}).$$

Proof. Since the projection π doesn't change geometry, it follows that near $x \in \pi^{-1}(b)$:

$$D_\theta(\pi^* G_b) = D_\theta(G_b),$$

where the left hand side is over \widetilde{M} and the right hand side is over M . For $b \in M$ and $x \in \pi^{-1}(b)$, let δ_x be the Dirac delta function with pole at x . Therefore,

$$D_\theta(\pi^* G_b) = \sum_{x \in \pi^{-1}(b)} \delta_x,$$

so π^*G_b has poles precisely in the set $\pi^{-1}(b) \subset \widetilde{M}$. For each fixed $p \in \pi^{-1}(b)$,

$$D_\theta(\overline{G}_p) = \delta_p.$$

Therefore $\pi^*G_b - \overline{G}_p$ is bounded near p . On the other hand, by the normalization $|\Phi'(p)| = 1$ and Φ is injective, we have

$$\overline{G}_p^{\frac{2}{n}}\theta = \Phi^*(H_q^{\frac{2}{n}}\theta_{S^{2n+1}}),$$

it follows that globally on $\widetilde{M} \setminus p$

$$(5.2) \quad ((C_q \circ \Phi)^{-1})^*(\overline{G}_p^{\frac{2}{n}}\theta) = \Theta.$$

From equations (5.1) and (5.2) and that π^*G_b and \overline{G}_p have the same pole strength near p ,

$$h(\infty) = \lim_{(z,t) \rightarrow \infty} h(z,t) = 1.$$

On the other hand, because \overline{G}_p is the minimal Green's function for D_θ on \widetilde{M} , Bony's strong maximum principle implies that globally on \widetilde{M} :

$$\pi^*G_b > \overline{G}_p,$$

which by equations (5.1) and (5.2) that for (z,t) near ∞ , we have $h = h(z,t) > 1$.

Next, we would like to show the asymptotic expansion of the function $h = h(z,t)$ near ∞ . By the transformation rule of the CR invariant sublaplacian, we have $R(G_b^{\frac{2}{n}}\theta) = 0$ globally on $M \setminus \{b\}$, where $R(G_b^{\frac{2}{n}}\theta)$ is the Webster curvature with respect to the contact form $G_b^{\frac{2}{n}}\theta$. Using the transformation rule again we have $D_\Theta(h) = 0$ for (z,t) in a neighborhood of $\infty \in H^n$. It follows that (since h is regular near ∞ and $h(\infty) = 1$):

$$h(z,t) = 1 + c_{-1} \cdot \rho^{-1} + \cdots + A_b \cdot \rho^{-2n} + O(\rho^{-2n-1}).$$

Consider the global CR inversion $\vartheta : H^n \setminus 0 \rightarrow H^n \setminus 0$ as follows:

$$\vartheta(z,t) = (\hat{z}, \hat{t}) = (-z/w, t/|w|^2),$$

where $w = t + i|z|^2$. Consider the following standard contact form

$$\Theta(z,t) = dt - i \sum_{\alpha=1}^n (z^\alpha \cdot dz^{\bar{\alpha}} - z^{\bar{\alpha}} \cdot dz^\alpha),$$

and

$$\Theta(\hat{z}, \hat{t}) = d\hat{t} - i \sum_{\alpha=1}^n (\hat{z}^\alpha \cdot d\hat{z}^{\bar{\alpha}} - \hat{z}^{\bar{\alpha}} \cdot d\hat{z}^\alpha).$$

It follows that $\rho(z, t)^2 = |w| = |\hat{w}|^{-1} = \rho(\hat{z}, \hat{t})^{-2}$ and

$$\Theta(\hat{z}, \hat{t}) = |w|^{-2} \cdot \Theta(z, t) = (\rho(\hat{z}, \hat{t})^{2n})^{\frac{2}{n}} \cdot \Theta(z, t).$$

Therefore,

$$h(z, t)^{\frac{2}{n}} \cdot \Theta(z, t) = (h(\hat{z}, \hat{t}) \cdot \rho(\hat{z}, \hat{t})^{-2n})^{\frac{2}{n}} \cdot \Theta(\hat{z}, \hat{t}).$$

By the facts that $R(h(z, t) \cdot \Theta(z, t)) = 0$ and $\Theta(\hat{z}, \hat{t})$ and the trasformation rule of the CR invariant sublaplacian, it follows that near but away from the origin in H^n , we have

$$D_{\Theta(\hat{z}, \hat{t})}(h(\hat{z}, \hat{t}) \cdot \rho(\hat{z}, \hat{t})^{-2n}) = 0, \text{ and } D_{\Theta(\hat{z}, \hat{t})}\rho(\hat{z}, \hat{t})^{-2n} = 0.$$

Therefore, near but away from the origin in H^n :

$$D_{\Theta(\hat{z}, \hat{t})}(h(\hat{z}, \hat{t}) \cdot \rho(\hat{z}, \hat{t})^{-2n} - \rho(\hat{z}, \hat{t})^{-2n}) = 0$$

and note that

$$\begin{aligned} & h(\hat{z}, \hat{t}) \cdot \rho(\hat{z}, \hat{t})^{-2n} - \rho(\hat{z}, \hat{t})^{-2n} \\ &= c_{-1} \cdot \rho(\hat{z}, \hat{t})^{-2n+1} + \cdots + c_{-2n+1} \cdot \rho(\hat{z}, \hat{t})^{-1} + A_b + O(\rho(\hat{z}, \hat{t})). \end{aligned}$$

A standard removable singularity argument (Proposition 5.17 in [21]) implies that

$$c_{-1} = \cdots = c_{-2n+1} = 0.$$

Therefore, we have the following aymptotic expansion of $h = h(z, t)$ near ∞ :

$$h(z, t) = 1 + A_b \cdot \rho(z, t)^{-2n} + O(\rho(z, t)^{-2n-1}).$$

This completes the lemma. \square

DEFINITION 5.1. We call the constant A_b CR mass.

We would like to remark that the constant A_b doesn't depend on the choice of local coordinates near $b \in M$ (see [23]). Corollary C states the positivity of the CR mass.

Proof. (of Corollary C)

By Theorem A, Φ is injective. It follows from Lemma 5.1 that for (z, t) near ∞

$$h = h(z, t) = 1 + A_b \cdot \rho(z, t)^{-2n} + O(\rho(z, t)^{-2n-1}) > 1.$$

Let $B_L(0)$ be a ball on H^n centered at 0 and with radius L such that

$$m = \min_{\partial B_L(0)} (h - 1) > 0.$$

Since in $H^n \setminus B_L(0)$,

$$D_{\Theta}(h - 1) = D_{\Theta}(mL^{2n}\rho^{-2n}) = 0$$

and

$$(h - 1)|_{\partial B_L(0)} \geq mL^{2n}\rho^{-2n}|_{\partial B_L(0)},$$

we conclude from Bony's maximum principle that in $H^n \setminus B_L(0)$,

$$h - 1 \geq mL^{2n}\rho^{-2n}.$$

Therefore, $A_b \geq mL^{2n} > 0$. \square

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